Gamow functionals on operator algebras

M Castagnino[1], M Gadella[2], R Id Betán[1] and R Laura[1]

[1] Facultad de Ciencias Exactas, Ingeniería y Agrimensura, Av Pellegrini 250, 2000 Rosario, Argentina
[2] Departamento de Física Teórica, Facultad de Ciencias, c./ Real de Burgos, sn 47011 Valladolid, Spain

Abstract
We obtain the precise form of two Gamow functionals representing the exponentially decaying part of a quantum resonance and its mirror image that grows exponentially, as a linear, positive and continuous functional on an algebra containing observables. These functionals do not admit normalization and, with an appropriate choice of the algebra, are time reversal of each other.

PACS numbers: 03.65.Db, 02.30.Sa, 02.30.Tb

1. Introduction

The goal of the present paper is to give a precise definition of the Gamow functional on a formalism that has been used previously to discuss a variety of topics such as resonance behaviour, decoherence, generalized states with diagonal singularity, etc [1–5]. This formalism has been inspired in previous work by Prigogine and collaborators [6–8].

Gamow vectors [9] are generalized eigenvectors of the total Hamiltonian, in a resonant scattering process, with complex eigenvalues given by the simple poles of the analytic continuation of the S-matrix [10] or the reduced resolvent [11–14]. As the Hamiltonian is a self-adjoint operator, its eigenvectors with complex eigenvalues cannot live in a Hilbert space but on certain extensions of the Hilbert spaces: the rigged Hilbert spaces (RHS) [10, 12, 13, 15]. Gamow vectors represent the exponentially decaying part of a resonance (for a discussion on the decay in quantum mechanics, see [16]). The question arises that whether a Gamow vector represents a truly quantum state, i.e. an element of the physical reality.

In conventional quantum mechanics in Hilbert space, let $|\varphi\rangle$ be a pure state; its corresponding density operator is given by $\rho = |\varphi\rangle\langle\varphi|$. The operator ρ represents the state $|\varphi\rangle$ in the Liouville space and, therefore, this is the object that should represent the state in quantum statistical mechanics. Thus, if we accept that the Gamow vector represents a quantum state, it must have its counterpart in quantum statistical mechanics. Since the Gamow vector belongs to an extension of the Hilbert space, its corresponding density matrix should belong to an extension of the conventional Liouville space, called the rigged Liouville space

(RLS) [17]. Although we can construct rigorously a dyadic product of Gamow vectors in the RLS, these objects do not satisfy the minimal requirements to be a state. In particular, objects like $\mathrm{tr}\,(|f_0\rangle\langle f_0|)$ or $\mathrm{tr}\,(|f_0\rangle\langle f_0|H)$, where $|f_0\rangle$ is the Gamow vector, are not defined. In other words, objects like $\langle f_0|f_0\rangle$ and $\langle f_0|H|f_0\rangle$, representing, respectively, the normalization and the mean value of the energy of a Gamow vector, cannot be defined. We have studied the properties of Gamow dyads in RLS in [18].

In statistical mechanics, states are also represented by continuous positive and normalized functionals on an algebra of observables [19]. This is the approach we wish to analyse in this paper. We shall construct an algebra of observables in which the Gamow 'state' can be defined as a continuous functional on this algebra. This functional is characterized by its decay mode and is also positive, but cannot be normalized (as its normalization results to be zero). Worse of all, the expectation values of the integer powers of the Hamiltonian, H^n, $n = 0, 1, 2, \ldots$, vanish. As a result of this discussion we conclude that the Gamow functional cannot represent a quantum state even if we admit the existence of particles with a purely exponential decaying mode.

This approach does not restrict its interest to statistical mechanics but is also suitable for applications to the theory of decaying nuclei [20].

To better understand the notion of Gamow functional, we need to use the notion of rigged Hilbert space (RHS). An RHS is a triplet of spaces

$$\mathbf{\Phi} \subset \mathcal{H} \subset \mathbf{\Phi}^{\times}$$

where $\mathcal{H}$ is the Hilbert space of pure normalized states of a quantum system, $\mathbf{\Phi}$ is a space of test vectors (usually a space of functions called the space of test functions) with its own topology which is stronger (in the sense that has more open sets, less convergent sequences and that the canonical injection $i\colon \mathbf{\Phi} \longmapsto \mathcal{H}$, $i(\varphi) = \varphi$, is continuous). $\mathbf{\Phi}^{\times}$ is the antidual of $\mathbf{\Phi}$ or the space of all continuous antilinear[1] functionals from $\mathbf{\Phi}$ to $\mathbb{C}$. It is precisely this extension $\mathbf{\Phi}^{\times}$ of the Hilbert space which allows the existence of generalized eigenvectors of an observable [15].

This paper is organized as follows. In section 2, we define the algebra of observables compatible with the 'free' or unperturbed Hamiltonian H_0. In section 3, we define the notion of states as functionals over this algebra. In section 4, we define the algebras of observables compatible with the total Hamiltonian H and the Gamow functionals on it. We can define these algebras in various ways and, with an appropriate definition of the algebras, the Gamow functionals are time reversal of each other. We conclude the paper with a mathematical appendix, in which we study the mathematical tools used in our study.

2. The algebra $\mathcal{A}_0$ of observables

The most intuitive model that produces quantum resonances is possibly the resonant scattering model, in which we assume the existence of a resonant scattering process [10] with two dynamics. The unperturbed or free dynamics is given by H_0 and the perturbed dynamics by $H := H_0 + V$. We assume also that the Møller wave operators exist and that the scattering is asymptotically complete [21]. In this case a theorem by Gelfand [22] and Maurin [23] states that there exists a complete set of generalized eigenvectors of H_0 (in a suitable RHS),

---

[1] A functional F on $\mathbf{\Phi}$ is antilinear, if it is a mapping from $\mathbf{\Phi}$ into $\mathbb{C}$ with the following condition

$$F(\alpha\,\varphi + \beta\,\psi) = \alpha^*\,F(\varphi) + \beta^*\,F(\psi)$$

where the star denotes complex conjugation.

$|E\rangle$, for all E in the continuous spectrum of H_0 (which we assume to be simple and equal to $\mathbb{R}^+ := [0, \infty)$):

$$H_0|E\rangle = E|E\rangle \qquad E \in \mathbb{R}^+.$$

The vector $|E\rangle$ belongs to the dual space $\Phi^\times$ of an RHS, $\Phi \subset \mathcal{H} \subset \Phi^\times$ and the completeness means that

$$H_0 = \int_0^\infty dE\, E|E\rangle\langle E| = \int_0^\infty dE \int_0^\infty dE'\, \delta(E - E')\, E|E\rangle\langle E'|. \tag{1}$$

Therefore, the expression (1) for H_0 means that $H_0 \in \mathcal{L}(\Phi, \Phi^\times)$, i.e. the space of continuous linear operators from Φ into $\Phi^\times$; see also [18]. The action of $|E\rangle$ on the test function $\varphi \in \Phi$ gives $[\varphi(E)]^*$, the complex conjugate of the value of φ at E. We also have that $\langle E|\varphi\rangle = \langle\varphi|E\rangle^*$.

Equation (1) allows us to obtain, at least formally, the following matrix element:

$$\langle E'|H_0|E''\rangle = \int_0^\infty dE\, E\langle E'|E\rangle\langle E|E''\rangle. \tag{2}$$

Since $\langle E'|E\rangle = \delta(E - E')$, where the deltas are relative to the integration from 0 to ∞, (2) is equal to $E'\delta(E' - E'')$ and, therefore, it is well defined as a distributional kernel.

Definition . *An operator O is said to be compatible with H_0 if it has the form*

$$O = \int_0^\infty dE\, O_E|E\rangle\langle E| + \int_0^\infty dE \int_0^\infty dE'\, O_{EE'}\, |E\rangle\langle E'| \tag{3}$$

where O_E and $O_{EE'}$ are ordinary functions[2] on the variables E and E' (see appendix). Here, the function O_E is an entire analytic function in a class[3] that contains polynomials in E. The function $O_{EE'}$ should be of the form

$$O_{EE'} = \sum_{ij} \lambda_{ij}\, \psi_i(E)\, \phi_j(E') \tag{4}$$

where $\psi_i(E)\, \phi_j(E') \in \mathcal{Z}$, i.e. are entire analytic functions on the variables E and E' (see appendix for a definition of $\mathcal{Z}$). As we see later, this is not the only possible choice for the functions $O_{EE'}$, although it must be, in any case functions on the complex variables E and E'. The sum in (4) is finite.

It is important to remark that the set of observables compatible with H_0 is an algebra, which we denote as $\mathcal{A}_0$. See appendix for the definition on the algebra operations on $\mathcal{A}_0$.

At this point it would be convenient to justify our choice. In fact, we want the following properties for the set of observables $\mathcal{A}_0$, compatible with H_0:

 (i) $\mathcal{A}_0$ should be an algebra. This permits the use of the traditional point of view according to which observables form a (topological) algebra and states are continuous, positive and normalizable functionals on this algebra [19].
(ii) The precise choice of $\mathcal{A}_0$ is largely a matter of convenience. First of all, the set of states must contain those which are physically meaningful. All the other criteria, do not seem to be very essential from the physical point of view.

   For instance: What kind of observables should we include in $\mathcal{A}_0$? Should functions on H_0, including H_0 itself, be included in $\mathcal{A}_0$?

   Although at the first sight one is tempted to give a positive answer to this question, we should notice that we want to discuss the nature of Gamow objects. These Gamow objects

---

[2] Here we are using the notation in [1–4, 20].
[3] This class is the sum $\mathcal{P} + \mathcal{Z}$ of the space $\mathcal{P}$ of the polynomials, considered as entire analytic functions of a complex variable, plus the space $\mathcal{Z}$ of entire analytic functions introduced in the appendix.

are supposed to describe an aspect of resonance behaviour and resonances are assumed to be produced in resonant scattering [10]. But then, our question not always has a positive answer in scattering theory. For instance, in the algebraic theory of scattering developed by Amrein *et al* [24], the algebra $\mathcal{A}_0$ contains only bounded operators in the bicommutant (operators which commute with those commuting with H_0) of H_0. Since H_0 is not bounded, H_0 is not in the $\mathcal{A}_0$ of [24].

(iii) What is really relevant here is that the algebra of observables be spanned by the dyads of the form $|E\rangle\langle E|$ and $|E\rangle\langle E'|$, where E and E' run out the continuous spectrum of H_0. To see this, at least intuitively, let us note that for a pair of state vectors ψ, φ and an observable O, we have

$$\langle\psi|O|\varphi\rangle = \int \langle\psi|E\rangle\langle E|O|E'\rangle\langle E|\varphi\rangle \, \mathrm{d}E.$$

Then, if the kernel $\langle E|O|E'\rangle$ satisfies the van Hove hypothesis[4] [5, 6],

$$\langle E|O|E'\rangle = O_E \, \delta(E - E') + O_{EE'}$$

we have that

$$\langle\psi|O|\varphi\rangle = \int \langle\psi|E\rangle\langle E|\varphi\rangle \, O_E \, \mathrm{d}E + \int \langle\psi|E\rangle\langle E'|\varphi\rangle \, O_{EE'} \, \mathrm{d}E \, \mathrm{d}E'$$

from where (4) follows.

Then the choice of the functions O_E and $O_{EE'}$ gives the observables that we want to consider.

(iv) As we shall see in the next section, we want to include in the formalism states which are outside the Hilbert–Schmidt space (and therefore are not density operators on the Hilbert space) and we have to adapt the algebra so as to include these singular objects.

(v) Since we want Gamow objects that are continuous functionals on operator algebras (not on $\mathcal{A}_0$ but instead on the derived algebras $\mathcal{A}_\pm$ to be defined in section 4) and since Gamow functionals are characterized by certain complex numbers (of the kind $E_R - \mathrm{i}\Gamma/2$, where E_R is the resonant energy and Γ the width [10]), it seems reasonable that the functions $O_{EE'}$ be defined over a complex domain. Analyticity of these functions over this domain will then allow us to perform all kinds of operations that are customary in the study of resonances and Gamow vectors: contour integrals, calculus of residues, etc. [10, 12, 15, 20].

(vi) The issue whether the algebra $\mathcal{A}_0$ (as well as the algebras $\mathcal{A}_\pm$ to be defined in section 4) has a precise physical meaning has the same answer as a similar question that has been addressed by the RHS: Given an RHS $\Phi \subset \mathcal{H} \subset \Phi^\times$, what is the physical meaning of the space of test vectors Φ? Should Φ be contained or even be spanned by the space of pure states which are physically preparable? Not neccesarily, for if Φ is dense in $\mathcal{H}$, any physically preparable state can be approached by a vector in Φ as much as we want, with respect to the norm of $\mathcal{H}$. This norm is produced by the scalar product, what gives the transition amplitudes. As a matter of fact, the space Φ is chosen for topological convenience as well as to determine the size of the dual space $\Phi^\times$, which must contain all generalized states (like plane waves). Thus, the specific form of the algebra $\mathcal{A}_0$ is also determined by mathematical convenience.

Once we have motivated the choice of $\mathcal{A}_0$, let us comment some of its properties.

It is interesting to note that the operator O commutes, according to the definition of the product in the algebra given in the appendix, with H_0 if and only if $O_{EE'} = 0$. The proof of this statement is also presented in the appendix.

---

[4] This hypothesis was introduced by van Hove in his study of unstable quantum systems.

In the appendix, we have also given the topology on the algebra $\mathcal{A}_0$ that will allow us to define continuous functionals on $\mathcal{A}_0$. We want to add that this topology makes the following mappings continuous:

$$O \longmapsto O_E \qquad O \longmapsto O_{EE'} \tag{5}$$

for all E, $E' \in \mathbb{C}$, where $\mathbb{C}$ is the complex plane. According to a useful notation [6], we can represent these two functionals as $(E|$ and $(EE'|$, respectively, so that

$$O_E = (E|O) \qquad O_{EE'} = (EE'|O) \tag{6}$$

which yields

$$O = \int_0^\infty (E|O)|E\rangle\langle E|\, \mathrm{d}E + \int_0^\infty \mathrm{d}E \int_0^\infty \mathrm{d}E' (EE'|O)\, |E\rangle\langle E'|. \tag{7}$$

This notation is consistent with [6]

$$|E\rangle\langle E| \equiv |E) \qquad |E\rangle\langle E'| \equiv |EE') \tag{8}$$

and

$$(E|w) = \delta(E - w) \qquad (EE'|ww') = \delta(E - w)\,\delta(E' - w'). \tag{9}$$

Taking into account that $\langle E|E'\rangle = \delta(E - E')$, where the delta refers to integration from 0 to ∞, we also obtain that

$$\langle E|O|E'\rangle = O_E\,\delta(E - E') + O_{EE'}. \tag{10}$$

It is also important to remark that only self-adjoint elements of $\mathcal{A}_0$ should be considered as observables. The condition for self-adjointness in our case is very simple. The formal adjoint of O is given by

$$O^\dagger := \int_0^\infty \mathrm{d}E\, O_E^* \,|E\rangle\langle E| + \int_0^\infty \mathrm{d}E \int_0^\infty \mathrm{d}E'\, O_{EE'}^* \,|E'\rangle\langle E|. \tag{11}$$

It is easy to show that this definition is consistent with the formula $(\varphi, O\psi) = (O^\dagger\varphi, \psi)$, when φ, $\psi \in \Phi$ and $(-, -)$ is the scalar product on the Hilbert space $\mathcal{H}$ (see (14)). Here, Φ is the space of test functions introduced earlier, on which $|E\rangle$ applies.

Definition . *We say that O is self-adjoint if $O = O^\dagger$. An operator O of the form (3) is an observable if and only if it is self-adjoint.*

Proposition 1. *The operator O is an observable if and only if*

$$O_E = O_E^* \qquad and \qquad O_{EE'} = O_{E'E}^* \tag{12}$$

where '$*$' means complex conjugate.

Proof. Let us assume that O is an observable. Then, it is immediate to show that

$$\langle E|O|E'\rangle = O_E\,\delta(E - E') + O_{EE'}$$
$$\langle E|O^\dagger|E'\rangle = O_E^*\,\delta(E - E') + O_{E'E}^*. \tag{13}$$

Since $O = O^\dagger$, (13) implies that $O_E = O_E^*$ and $O_{EE'} = O_{E'E}^*$. Reciprocally, if these two equations hold, then, for any pair of test vectors φ and ψ, we have that $(\varphi, O\psi) = (O^\dagger\varphi, \psi)$, as we can easily check. Observe that $O_{EE'}$ is complex in general. $\qquad\qquad\square$

Now, we are more interested in clarifying the formalism we use here and the role of quantum states on it. We do this in the next section.

3. States

The theorem of Gelfand and Maurin [22, 23] establishes the existence of an RHS $\Phi \subset \mathcal{H} \subset \Phi^\times$ such that if $\psi, \varphi \in \Phi$, we have

$$(\psi, \varphi) = \int_0^\infty \langle \psi | E \rangle \langle E | \varphi \rangle \, dE \tag{14}$$

where the brackets (-,-) denote scalar product on the Hilbert space $\mathcal{H}$. If we omit the arbitrary vector $\psi \in \Phi$ in (14), we have

$$\varphi = \int_0^\infty |E\rangle \langle E | \varphi \rangle \, dE. \tag{15}$$

However, formula (15) is inconsistent as far as its right-hand side is a functional on Φ (and therefore a vector in $\Phi^\times$) and its left-hand side is a vector in Φ. As $\Phi \subset \Phi^\times$, φ can be also looked as a vector in $\Phi^\times$. For convenience, we introduce the identity mapping I that maps a vector on Φ as the same vector as member of $\Phi^\times$. This identity can be written as

$$I = \int_0^\infty |E\rangle \langle E| \, dE. \tag{16}$$

At this point, we can start the discussion on states by calculating the mean value of a pure state ψ, considered as a vector with norm one on the Hilbert space $\mathcal{H}$, on the observable O. This is given by

$$\begin{aligned}
\langle \psi | O | \psi \rangle &= \left[\int_0^\infty dE \, \langle \psi | E \rangle \langle E | \right] \left[\int_0^\infty dE' \, O_{E'} \, |E'\rangle \langle E' | \right. \\
&\quad \left. + \int_0^\infty dE' \int_0^\infty dE'' \, O_{E'E''} \, |E'\rangle \langle E'' | \right] \left[\int_0^\infty dE''' \, |E'''\rangle \langle E''' | \psi \rangle \right] \\
&= \int_0^\infty dE \, |\langle \psi | E \rangle|^2 \, O_E + \int_0^\infty dE \int_0^\infty dE' \, O_{EE'} \, \langle \psi | E \rangle \langle E' | \psi \rangle \\
&= \int_0^\infty dE \, |\psi(E)|^2 \, O_E + \int_0^\infty dE \int_0^\infty dE' \, O_{EE'} \, \psi^*(E) \, \psi(E').
\end{aligned} \tag{17}$$

Obviously, this comes after $\langle \varepsilon | \zeta \rangle = \delta(\varepsilon - \zeta)$, when $\varepsilon, \zeta = E, E', E'', E'''$. We can use here the notation $\rho_E = |\psi(E)|^2$ and $\rho_{EE'} = \psi^*(E) \psi(E')$. Note that $\rho_E = \rho_{EE}$.

Now, let ρ be a mixture of states. Then, $\rho = \sum_i \lambda_i |\psi_i\rangle \langle \psi_i|$ with $\sum_i \lambda_i = 1$, $\lambda_i \geqslant 0$ and $\langle \psi_i | \psi_j \rangle = \delta_{ij}$. The mean value of the observable O, compatible with H_0, in the state ρ is given by

$$\begin{aligned}
\operatorname{tr} \rho O &= \sum_i \lambda_i \, \langle \psi_i | O | \psi_i \rangle \\
&= \sum_i \lambda_i \int_0^\infty dE \, |\psi_i(E)|^2 \, O_E + \sum_i \lambda_i \int_0^\infty dE \int_0^\infty dE' \, O_{EE'} \, \psi_i^*(E) \, \psi_i(E') \\
&= \int_0^\infty dE \left[\sum_i \lambda_i \, |\psi_i(E)|^2 \right] O_E \\
&\quad + \int_0^\infty dE \int_0^\infty dE' \, O_{EE'} \left[\sum_i \lambda_i \psi_i^*(E) \, \psi_i(E') \right].
\end{aligned} \tag{18}$$

We call $\rho_E := \sum_i \lambda_i \, |\psi_i(E)|^2$ and $\rho_{EE'} := \sum_i \lambda_i \, \psi_i^*(E) \, \psi_i(E')$. Note that ρ_E is real and $\rho_{EE'}$ is complex in general. It is also true that $\rho_E = \rho_{EE}$.

Now, observe that in both cases we can write the state as

$$\rho = \int_0^\infty dE \, \rho_E \, (E| + \int_0^\infty dE \int_0^\infty dE' \, \rho_{EE'} \, (EE'| \qquad (19)$$

so that when applied to the observable O written as

$$O = \int_0^\infty dE \, O_E \, |E) + \int_0^\infty dE \int_0^\infty dE' \, O_{EE'} |EE') \qquad (20)$$

gives the result

$$(\rho|O) := \operatorname{tr} \rho O = \int_0^\infty dE \, \rho_E \, O_E + \int_0^\infty dE \int_0^\infty dE' \, \rho_{EE'} \, O_{EE'} \qquad (21)$$

where we have used the relations (9). For the two choices (17) and (18), ρ in (19) defines a continuous positive and normalized functional on $\mathcal{A}_0$ and therefore a state.

At this point, we observe that the algebra $\mathcal{A}_0$ is a direct sum of two subalgebras, the algebra $\mathfrak{B}$ spanned by

$$\int_0^\infty dE \, O_E |E) \qquad (|E) = |E\rangle\langle E|)$$

and the algebra $\mathfrak{C}$ spanned by

$$\int_0^\infty dE \int_0^\infty dE' \, O_{EE'} |EE') \qquad (|EE') = |E\rangle\langle E'|).$$

Both algebras do not have common elements other than the zero (see appendix). Therefore $\mathcal{A}_0 = \mathfrak{B} + \mathfrak{C}$ is a direct sum. As a consequence, every continuous linear functionals on $\mathcal{A}_0$ is the sum of a continuous linear functional on $\mathfrak{B}$ plus a continuous linear functional on $\mathfrak{C}$.

The algebras $\mathfrak{B}$ and $\mathfrak{C}$ are, respectively, isomorphic to the algebras of the functions of the form O_E and $O_{EE'}$. Therefore, $\mathcal{A}_0$ is isomorphic to the algebra of pairs of functions $(O_E, O_{EE'})$ with a product that can be immediately obtained from the product on $\mathcal{A}_0$.

From all this, we conclude that the most general form of a state on $\mathcal{A}_0$ is of the form (19) being ρ_E and $\rho_{EE'}$ continuous linear functionals (distributions) on the spaces of functions of the form O_E and $O_{EE'}$, respectively (see appendix).

In this formalism, we see that there are three kinds of states:

(i) Pure states. A state is pure if and only if there is a square integrable function $\psi(E)$ such that $\rho_E = |\psi(E)|^2$ and $\rho_{EE'} = \psi^*(E)\psi(E')$.
(ii) Mixtures. For mixtures $\rho_{EE} = \rho_E$.
(iii) Generalized states, which are all others.

Remarks

(i) Pure states and mixtures have the property that $\rho_E = \rho_{EE}$. The converse is also true, if ρ_{EE} is well defined and $\rho_E = \rho_{EE}$, then (19) represents either a pure state or a mixture, i.e., it admits a representation as a density operator on Hilbert space. On the other hand, generalized states cannot be represented as a density operator on a Hilbert space. The need for generalized states has been established by van Hove first [5] and a mathematically consistent definition of them was given in [6]. Our formalism is clearly inspired in [6], although our goals are different as we try to understand the role of the Gamow objects on it.
(ii) There are two kinds of generalized states, those for which $\rho(E, E)$ is well defined in a distributional sense and those for which it is not. For example, assume that $\rho(E, E') = \delta(E - E_0)\delta(E' - E_0)$. In this case, obviously $\rho(E, E)$ does not make

sense. If for a given state $\rho(E, E)$ is well defined, this is a generalized state if and only if $\rho(E) \neq \rho(E, E)$.

The evolution of the state ρ under the free Hamiltonian H_0 is

$$(\rho_t | O) = \left(\rho_0 \mid e^{itH_0} \, O \, e^{-itH_0}\right)$$

$$= \int_0^\infty dE \, O_E \rho_E + \int_0^\infty dE \int_0^\infty dE' \, O_{EE'} \, e^{it(E-E')} \, \rho_{EE'}. \qquad (22)$$

If $O_{EE'}$ is bounded and ρ is a mixture, due to the integrability of $\sum_i \lambda_i \, \psi^*(E)\psi(E')$, then $O_{EE'}\rho_{EE'}$ is also integrable and the second integral term in (22) vanishes as $t \longmapsto \infty$ as the result of the Riemann–Lebesgue lemma. After the limit process, only the first term remains. This fact is usually called decoherence.

4. The algebras $\mathcal{A}_\pm$ of observables

The algebras $\mathcal{A}_\pm$ play the same role with respect to the total Hamiltonian H as the algebra $\mathcal{A}_0$ with respect to H_0.

First of all, let $\Omega_\pm$ be the Møller wave operators, defined customary as [21]:

$$\Omega_+\varphi = \lim_{t \mapsto +\infty} e^{itH} \, e^{-itH_0} \, \varphi = \varphi^+$$

and

$$\Omega_-\varphi = \lim_{t \mapsto -\infty} e^{itH} \, e^{-itH_0} \, \varphi = \varphi^-$$

whenever these limits exist. The Møller wave operators relate state vectors which evolve with the total Hamiltonian H with state vectors which evolve with the free Hamiltonian H_0 and that are asymptotically (as $t \longmapsto \pm\infty$) identical (in our case φ evolves freely and $\varphi^\pm$ with H and $\lim_{t \mapsto \pm\infty} \left(e^{-itH_0}\varphi - e^{-itH}\varphi^\pm\right) = 0$).

As the Møller wave operators are assumed to exist, let us define [15][5]

$$|E^\pm\rangle = \Omega_\pm|E\rangle. \qquad (23)$$

The definition (23) makes sense as proven in [15]. Now, take O as in (3) and write

$$O^\pm = \Omega_\pm \, O\Omega_\pm^\dagger. \qquad (24)$$

Since (23) implies[6] that $\langle E|\Omega_\pm^\dagger = \langle E^\pm|$, the operators $O^\pm$ can then be written as

$$O^\pm = \int_0^\infty O_E|E^\pm\rangle\langle E^\pm| \, dE + \int_0^\infty dE \int_0^\infty dE' \, |E^\pm\rangle\langle E'^\pm| \, O_{EE'}. \qquad (25)$$

[5] If we define $\Phi^\pm := \Omega_\pm\Phi$, we have two new triplets

$$\Phi^\pm \subset \mathcal{H} \subset (\Phi^\pm)^\times$$

where $\mathcal{H}$ is the absolutely continuous part of the Hilbert space with respect to H (see [21]). The Møller operators can be extended to bicontinuous mappings from $\Phi^\times$ into $(\Phi^\pm)^\times$, so that (23) makes sense. This definition is made through the duality formula

$$\langle \varphi|E \rangle = \langle \Omega_\pm\varphi|\Omega_\pm|E \rangle = \langle \varphi^\pm|E^\pm \rangle$$

where φ is an arbitrary vector in Φ.

[6] To see this, write $\langle E|\varphi \rangle := \langle \varphi|E \rangle^*$ with $\varphi \in \Phi$. Then

$$\langle E^\pm|\varphi^\pm \rangle = \langle E|\varphi \rangle = \langle E|\Omega_\pm^\dagger\Omega_\pm|\varphi \rangle = \langle E|\Omega_\pm^\dagger|\varphi^\pm \rangle.$$

This expression is valid for any $\varphi^\pm \in \Phi^\pm$. Then $\langle E|\Omega_\pm^\dagger = \langle E^\pm|$ follows.

We say that an operator[7] $O^\pm$ is compatible with H if and only if, it can be written in the form given in equation (25). Since

$$\langle E^\pm | w^\pm \rangle = \langle E | \Omega_\pm^\dagger \, \Omega_\pm | w \rangle = \langle E | w \rangle = \delta(E - w) \tag{26}$$

we obtain that the operators of the type O^+ and O^- in (25) form respective algebras that we call $\mathcal{A}_+$ and $\mathcal{A}_-$ (see in the appendix how to define the product for two elements of $\mathcal{A}_0$. After (26), it is clear that the product in $\mathcal{A}_\pm$ is defined analogously). Since the operators $\Omega_\pm$ are unitary[8], the algebras

$$\mathcal{A}_\pm := \Omega_\pm \, \mathcal{A}_0 \Omega_\pm^\dagger$$

are isomorphic (algebraically and topologically) to the algebra $\mathcal{A}_0$.

States on these algebras have the form

$$\rho^\pm = \int_0^\infty \rho_E (E^\pm | \, \mathrm{d}E + \int_0^\infty \mathrm{d}E \int_0^\infty \mathrm{d}E' \, \rho_{EE'} \, (EE'^\pm | \tag{27}$$

where

$$(E^\pm | O^\pm) = O_E \qquad (EE'^\pm | O^\pm) = O_{EE'}. \tag{28}$$

The operators (25) can also be written as

$$O^\pm = \int_0^\infty \mathrm{d}E \, O_E \, | E^\pm) + \int_0^\infty \mathrm{d}E \int_0^\infty \mathrm{d}E' \, O_{EE'} \, | EE'^\pm) \tag{29}$$

so that

$$(E^\pm | w^\pm) = \delta(E - w) \qquad (EE'^\pm | ww'^\pm) = \delta(E - w) \, \delta(E' - w'). \tag{30}$$

This means that the operational rules in $\mathcal{A}_\pm$ are the same as in $\mathcal{A}_0$. The same can be said about the topology as the components O_E and $O_{EE'}$ of both algebras are the same. This topology is transported from $\mathcal{A}_0$ to $\mathcal{A}_\pm$ by the Møller operators. Also pure states, mixtures and generalized states with diagonal singularity can be written as functionals on $\mathcal{A}_\pm$ exactly as on $\mathcal{A}_0$. Time evolution of $\rho^\pm$ with respect to H is of the form

$$(\rho_t^\pm | O^\pm) = \int_0^\infty \mathrm{d}E \rho_E \, O_E + \int_0^\infty \mathrm{d}E \int_0^\infty \mathrm{d}E' \, \mathrm{e}^{\mathrm{it}(E - E')} \, \rho_{EE'} \, O_{EE'} \tag{31}$$

Observe that the first integral in (31) does not evolve in time. The second part vanishes for $t \longmapsto \pm\infty$ if $\rho_{EE'} \, O_{EE'}$ is an integrable function in the two dimensional variable (E, E').

5. The Gamow functionals

If the pair $\{H_0, H\}$ produces resonances, these are manifested as pairs of poles of the same multiplicity in the analytic continuation of the S-matrix in the energy representation [10] or the reduced resolvent [14]. Both are complex functions of the energy considered as a complex variable and, under very general conditions [14], have poles located at the same points. These poles may have arbitrary multiplicity and appear into complex conjugate pairs of the same multiplicity, although only simple resonance poles yield exponentially decaying Gamow vectors [25]. Thus, let us assume that we have a pair of resonance poles located at the points $z_0 = E_R - \mathrm{i}\Gamma/2$ and its complex conjugate z_0^*. Within the above formalism is quite easy to define the decaying Gamow functional.

---

[7] These operators are continuous linear mappings from $\Phi^\pm$ into $(\Phi^\pm)^\times$. Therefore, they are a generalization of the usual notion of operator as a linear mapping on $\mathcal{H}$.

[8] We assume asymptotic completeness [21]. Therefore, $\Omega_\pm$ are unitary operators between the absolutely continuous subspaces of H_0 and H.

For any function $\psi \in \mathcal{Z}$, the functional δ_z maps $\psi(E)$ into its value at z, $\psi(z)$. If ϕ is another function in $\mathcal{Z}$, the tensor product $\delta_z \otimes \delta_{z'}$ maps the function $\psi \otimes \phi$ into $\psi(z)\phi(z')$.

Then, we define the decaying Gamow functional as

$$\rho_D := \int_0^\infty dE \int_0^\infty dE' \, \delta_{z_0^*} \otimes \delta_{z_0} (EE'^+|. \tag{32}$$

This is obviously an element of $\mathcal{A}_+^\times$, the dual of the algebra $\mathcal{A}_+$. Note that $(\rho_D)_E = 0$ and $(\rho_D)_{EE'} = \delta_{z_0^*} \otimes \delta_{z_0}$. The action of ρ_D on $O \in \mathcal{A}_+$ is given by

$$(\rho_D|O) = O_{z_0^* z_0}. \tag{33}$$

The functional ρ_D has the following properties: $(\rho_D(0) = \rho_D)$

$$(\rho_D(t)|O) = \int_0^\infty dE \int_0^\infty dE' \, \delta_{z_0^* z_0} \, O_{EE'} \, e^{it(E-E')} = e^{it(z_0^*-z_0)} \, O_{z_0^* z_0}$$
$$= e^{-t\Gamma} \, O_{z_0^* z_0} = e^{-t\Gamma} \, (\rho_D|O) \tag{34}$$

where $z_0 = E_R - i\frac{\Gamma}{2}$, being E_R the resonant energy and Γ the mean life. We observe that ρ_D decays exponentially for all values of the time. Other properties of ρ_D are

$$(\rho_D|I^+) = 0 \tag{35}$$

where I^+ is given by

$$I^+ = \Omega_+ \, I \, \Omega_+^\dagger = \int_0^\infty |E^+\rangle\langle E^+| \, dE$$

and[9] I is given in (16)

$$(\rho_D|H^n) = 0, \qquad n = 0, 1, 2, \ldots \tag{36}$$

where

$$H^n = \int_0^\infty dE \, E^n \, |E^+).$$

We can choose the functions O_E and $O_{EE'}$ in such a way that the evolution $\rho_D(t)$ for the Gamow functional is either valid for $t > 0$ only or for all values of time. In the latter case, the evolution law is not given by a semigroup and this eliminates the problem of fixing the time $t = 0$ as 'the instant at which the preparation of the quasistationary state has been completed and starts to decay' [10, 26]. In the former case, $O_{EE'}$ cannot belong to a class of entire functions on the variables EE', as discussed later.

In summary, the Gamow functional ρ_D has the following properties:

1. It is linear and continuous on the algebra $\mathcal{A}_+$.
2. It is positive, i.e. $(\rho_D|(O^+)^\dagger O^+) \geqslant 0$.
3. Equation (35) shows that the functional ρ_D does not admit a normalization[10]. A quantum state is defined to be a linear functional on an algebra, containing the observables of the system, which is continuous, positive and normalizable. As ρ_D is not normalizable, it is not a state in the ordinary sense. In addition, equation (36) shows that the mean value of all powers of H on ρ_D vanish. This is another argument to conclude that ρ_D does not represent a truly quantum state.

---

[9] Observe that I^+ is the canonical injection from Φ_+ into $\Phi_+^\dagger$. See footnote 3.
[10] Should we have $(\rho_D|I^+) = \alpha \neq 0$, we could still normalize the functional as ρ_D/α.

Along the decaying Gamow functional there is the growing Gamow functional ρ_G which is defined on $\mathcal{A}_-$ as

$$\rho_G = \int_0^\infty \mathrm{d}E \int_0^\infty \mathrm{d}E' \, \delta_{z_0} \otimes \delta_{z_0^*} \, (EE'^-|. \tag{37}$$

The growing Gamow functional ρ_G has the following properties:

1. The mean value of O^- in ρ^G is given by

$$(\rho_G|O^-) = O_{z_0 z_0^*}. \tag{38}$$

2. It grows exponentially at all times

$$(\rho_G(t)|O^-) = \mathrm{e}^{t\Gamma} (\rho_G|O^-) \tag{39}$$

   with $\rho_G = \rho_G(0)$.

3. It is not normalizable

$$(\rho_G|I^-) = 0 \qquad I^- = \int_0^\infty |E^-\rangle\langle E^-| \, \mathrm{d}E. \tag{40}$$

4. The mean value of the energy on ρ_G is zero

$$(\rho_G|H^n) = 0, \qquad n = 0, 1, 2, \ldots. \tag{41}$$

The relation between the algebras $\mathcal{A}_+$ and $\mathcal{A}_-$ is given by the time reversal operator T. In fact , we have $T|E^\pm\rangle = |E^\mp\rangle$, $T\Phi^\pm = \Phi^\mp$ and $T|\phi^\pm\rangle = |\phi^\mp\rangle$ [27], so that

$$\langle E^\pm|T|\phi^\mp\rangle = \langle E^\pm|\phi^\pm\rangle = \langle E|\phi\rangle = \langle E^\mp|\phi^\mp\rangle \tag{42}$$

where $|\phi\rangle := \Omega_+^{-1}|\phi^+\rangle = \Omega_-^{-1}|\phi^-\rangle$ and $|E\rangle = \Omega_+^{-1}|E^+\rangle = \Omega_-^{-1}|E^-\rangle$ [15]. Therefore,

$$\langle E^\pm|T = \langle E^\mp|. \tag{43}$$

Therefore, if

$$O^\pm = \int_0^\infty \mathrm{d}E \, O_E \, |E^\pm\rangle\langle E^\pm| + \int_0^\infty \mathrm{d}E \int_0^\infty \mathrm{d}E' \, O_{EE'} \, |E^\pm\rangle\langle E'^\pm|$$

we have that

$$TO^\pm T = \int_0^\infty \mathrm{d}E \, O_E^* \, |E^\mp\rangle\langle E^\mp| + \int_0^\infty \mathrm{d}E \int_0^\infty \mathrm{d}E' \, O_{EE'}^* \, |E^\mp\rangle\langle E'^\mp| \tag{44}$$

(we recall that $T\alpha|\eta\rangle = \alpha^* T|\eta\rangle$). Thus, we obtain

$$\mathcal{A}_\pm = T\mathcal{A}_\mp T. \tag{45}$$

The relation (44) implies a relation between ρ_D and ρ_G, provided that we redefine the algebras $\mathcal{A}_\pm$. In the new $\mathcal{A}_\pm$ the functions O_E are now polynomials on the complex variable E. In the new algebras $\mathcal{A}_\pm$ the functions $O_{EE'}$ will be different for $\mathcal{A}_+$ and for $\mathcal{A}_-$. For $\mathcal{A}_+$, $O_{EE'}$ is of the form (4) with

$$\psi_i(E) \in S \cap \mathcal{H}_+^2 \qquad \phi_j(E') \in S \cap \mathcal{H}_-^2 \qquad i, j = 1, 2, \ldots \tag{46}$$

where S is the Schwartz space[11] and $\mathcal{H}^\pm$ are the spaces of Hardy functions on the upper half plane and the lower half plane. Hardy functions are analytic in their respective half planes and their boundary values on the real line are square integrable functions (see appendix). Thus, $O_{EE'} \in S \cap \mathcal{H}_+^2 \otimes S \cap \mathcal{H}_-^2$ in the algebraic sense.

---

[11] Functions in S are indefinitely differentiable at all points and they and their derivatives go to zero at $\pm\infty$ faster than the inverse of any polynomial.

For $\mathcal{A}_-$, O'_{EE} is of the form (4) with

$$\psi_i(E) \in S \cap \mathcal{H}^2_- \qquad \phi_j(E') \in S \cap \mathcal{H}^2_+ \qquad i, j = 1, 2, \ldots \tag{47}$$

Thus, $O_{EE'} \in S \cap \mathcal{H}^2_- \otimes S \cap \mathcal{H}^2_+$.

Nothing in the formalism presented so far changes with this choice except the topology of the algebras (plus the irrelevant fact that we now have two isomorphic $\mathcal{A}_0$ algebras; it is not necessary to insist in this point). However, this choice has an interesting property: the time reversal of ρ_D is ρ_G and vice versa.

Before discussing this interesting point, it is important to remark that if $O_{EE'} \in S \cap \mathcal{H}^2_+ \otimes S \cap \mathcal{H}^2_-$, then $\mathrm{e}^{\mathrm{i}t(E-E')} O_{EE'} \in S \cap \mathcal{H}^2_+ \otimes S \cap \mathcal{H}^2_-$ if and only if $t \geqslant 0$. The proof is given in the appendix. Analogously, if $O_{EE'} \in S \cap \mathcal{H}^2_- \otimes S \cap \mathcal{H}^2_+$, then $\mathrm{e}^{\mathrm{i}t(E-E')} O_{EE'} \in S \cap \mathcal{H}^2_- \otimes S \cap \mathcal{H}^2_+$ if and only if $t \leqslant 0$. Thus, the time evolution for ρ_D makes sense for $t \geqslant 0$ only and time evolution for ρ_G makes sense for $t \leqslant 0$ only. Exactly as it happens with the Gamow vectors defined in [15].

Let us come back to the time reversal of the Gamow functionals. For ρ_D the time reversal operation is defined as

$$(\rho_D^T | O^-) := (\rho_D | TO^-T). \tag{48}$$

Since

$$TO^-T = \int_0^\infty \mathrm{d}E \, O_E^* \, |E^+\rangle\langle E^+| + \int_0^\infty \mathrm{d}E \int_0^\infty \mathrm{d}E' \, O_{EE'}^* |E^+\rangle\langle E'^+| \tag{49}$$

we have

$$(\rho_D | TO^-T) = O_{z_0^* z_0}^*. \tag{50}$$

Observe that, with this new definition,

$$O_{EE'} = \sum_{ij} \varphi_i(E) \, \psi_j(E') \tag{51}$$

(the coefficients λ_{ij} in (4) can be absorbed by the functions $\varphi_i(E) \, \psi_j(E')$) with

$$\varphi_i(E) \in \mathcal{H}^2_- \cap S \qquad \psi_j(E') \in \mathcal{H}^2_+ \cap S. \tag{52}$$

After the properties of Hardy functions [33], we have that

$$\varphi_i^*(E) \in \mathcal{H}^2_+ \cap S \qquad \psi_j^*(E') \in \mathcal{H}^2_- \cap S \tag{53}$$

and[12]

$$\varphi_i^*(z^*) = \varphi_i(z) \qquad \psi_j^*(z) = \psi_j(z^*). \tag{54}$$

Thus,

$$O_{z_0^* z_0}^* = \sum_{ij} \varphi_i^*(z_0^*) \, \psi_j^*(z_0) = \sum_{ij} \varphi_i(z_0) \, \psi_j(z_0^*) = O_{z_0 z_0^*}. \tag{55}$$

We conclude that for arbitrary $O^- \in \mathcal{A}_-$ we have

$$(\rho_D^T | O^-) = O_{z_0 z_0^*} = (\rho_G | O^-). \tag{56}$$

Thus

$$\rho_D^T = \rho_G. \tag{57}$$

Analogously,

$$\rho_G^T = \rho_D. \tag{58}$$

---

[12] This property is not true in general if $\varphi_i(E), \psi_j(E') \in \mathcal{Z}$.

We observe that the decaying Gamow functional and its mirror image act on different algebras.

It is a belief that resonances are irreversible systems and also that there exists a microphysical arrow of time in processes like quantum decay [12, 28, 29]. This belief is expressed into mathematical form by choosing the test spaces $\Phi^{\pm}$ for the Gamow vectors so that time evolution is defined for the decaying Gamow vector $|f_0\rangle$ for $t \geqslant 0$ only [15]. With our second choice for the algebras $\mathcal{A}_{\pm}$ a similar situation occurs as the evolution group splits into two semigroups and therefore, this picture may also be valid as a mathematical formulation of irreversibility in decaying systems.

Acknowledgments

Appendix

This is a mathematical appendix in which we shall construct explicitly the algebras $\mathcal{A}_0$ and $\mathcal{A}_{\pm}$ and their topologies. Due to the simple relation between these algebras, it is sufficient to construct $\mathcal{A}_0$. For this we have two possibilities: either the functions O_E are entire analytic or are Hardy.

The former option is simpler and the construction is as follows: Let $\mathcal{D}$ be the space of infinitely differentiable complex functions on the set of real numbers that have compact support. The Fourier transform of a function in $\mathcal{D}$ is entire analytic [30]. Therefore, the space of Fourier transforms of $\mathcal{D}$, $\mathcal{Z} = \mathcal{F}(\mathcal{D})$, is a space of entire analytic functions. This space has its own topology [30, 32] and the product of two functions in $\mathcal{Z}$ is another function in $\mathcal{Z}$ [30]. Furthermore, the product of a polynomial $p(z)$ times $f(z) \in \mathcal{Z}$ also belongs to $\mathcal{Z}$, i.e. $p(z)f(z) \in \mathcal{Z}$ (which can easily derived from theorems 6.30 and 6.37 in [30]).

Then, O_E is a sum of a function in $\mathcal{Z}$ plus a polynomial. The two variable function, $O_{EE'}$, has the form (4) with $\varphi_i(E)$ and $\psi_j(E')$ in $\mathcal{Z}$. Then, $O_{EE'} \in \mathcal{Z} \otimes \mathcal{Z}$. To show that $\mathcal{A}_0$ is an algebra, let us write

$$G := \int_0^{\infty} dE \, G_E \, |E\rangle\langle E| + \int_0^{\infty} dE \int_0^{\infty} dE' \, G_{EE'} \, |E\rangle\langle E'|$$

where G_E and $G_{EE'}$ are as O_E and $O_{EE'}$. Then,

$$OG = \left\{ \int_0^{\infty} dE \, O_E \, |E\rangle\langle E| + \int_0^{\infty} dE \int_0^{\infty} dE' \, O_{EE'} \, |E\rangle\langle E'| \right\}$$

$$\times \left\{ \int_0^{\infty} dw \, G_w \, |w\rangle\langle w| + \int_0^{\infty} dw \int_0^{\infty} dw' \, G_{ww'} \, |w\rangle\langle w'| \right\}$$

$$= \int_0^{\infty} dE \int_0^{\infty} dw \, O_E \, G_w \, |E\rangle\langle E|w\rangle\langle w|$$

$$+ \int_0^{\infty} dE \int_0^{\infty} dw \int_0^{\infty} dw' \, O_E \, G_{ww'} \, |E\rangle\langle E|w\rangle\langle w'|$$

$$+ \int_0^{\infty} dE \int_0^{\infty} dE' \int_0^{\infty} dw \, O_{EE'} \, G_w \, |E\rangle\langle E'|w\rangle\langle w|$$

$$+ \int_0^\infty dE \int_0^\infty dE' \int_0^\infty dw \int_0^\infty dw' \, O_{EE'} \, G_{ww'} \, |E\rangle\langle E'|w\rangle\langle w'|$$

$$= \int_0^\infty dE \, O_E \, G_E \, |E\rangle\langle E| + \int_0^\infty dE \int_0^\infty dE' O_E \, G_{EE'} \, |E\rangle\langle E'|$$

$$+ \int_0^\infty dE \int_0^\infty dE' \, G_{E'} \, O_{EE'} \, |E\rangle\langle E'|$$

$$+ \int_0^\infty dE \int_0^\infty dE' \int_0^\infty dw' \, O_{EE'} \, G_{E'w'} \, |E\rangle\langle w'|. \tag{59}$$

Now, $O_E G_E$ is either a polynomial on E or a function in $\mathcal{Z}$. The functions $O_E \, G_{EE'}$ and $G_{E'} \, O_{EE'}$ are of the form (4). Let us take the last integral in (43) and interchange E' and w' on it. We have

$$\int_0^\infty dE \int_0^\infty dE' \, |E\rangle\langle E'| \int_0^\infty dw' O_{Ew'} \, G_{w'E'}. \tag{60}$$

We can immediate see that the last integral in (60) is a function of the form (4). This shows that $\mathcal{A}_0$ is an algebra. In order to define a topology on this algebra, we first note that $\mathcal{A}_0$ considered as a vector space is the direct sum of three spaces

$$\mathcal{P} + \mathcal{Z} + \mathcal{Z} \otimes \mathcal{Z} \tag{61}$$

where $\mathcal{P}$ is the space of polynomials on the complex variable E. Let us topologize $\mathcal{P}$ as follows: consider the space of all functions $f(E) \in L^2[0, \infty)$ such that

$$\int_0^\infty |p(E) \, f(E)|^2 \, dE < \infty. \tag{62}$$

This space is dense in $L^2[0, \infty)$. For each function $f(E)$ of this kind, we define on $\mathcal{P}$ the following seminorm:

$$q_{f,K}(p) := \sqrt{\int_0^\infty |p(E) \, f(E)|^2 \, dE} + \sup_{E \in K} |p(E)| \qquad \forall p \in \mathcal{P} \tag{63}$$

K being a compact set in $\mathbb{C}$.

The topologies in $\mathcal{Z}$ [30] and in $\mathcal{Z} \otimes \mathcal{Z}$ [31] are standard, so that for any $p(E) + O_E + O_{EE'} \in \mathcal{P} + \mathcal{Z} + \mathcal{Z} \otimes \mathcal{Z}$, a typical seminorm π is of the form

$$\pi(p(E) + O_E + O_{EE'}) = q_{f,K}(p) + q(O_E) + r(O_{EE'}) \tag{64}$$

where q is a seminorm in $\mathcal{Z}$ and r a seminorm in $\mathcal{Z} \otimes \mathcal{Z}$.

Observe that not all quantum pure states are now allowed but only those satisfying (63). This is quite natural as condition (63) is fulfilled by the states in the domain of H^n, $n = 0, 1, 2, \ldots$ only. A similar restriction is required for mixtures.

Now, the topology on the algebras $\mathcal{A}_\pm$ goes exactly as for $\mathcal{A}_0$, since these algebras are isomorphic by construction.

Functionals as $(E^+|$, $(EE'^+|$ and ρ_D are continuous in $\mathcal{A}_+$ as $(E^-|$, $(EE'^-|$ and ρ_G are continuous in $\mathcal{A}_-$. The proof is technical and we omit it here.

The second possibility for the algebras $\mathcal{A}_\pm$ has been already presented (see formulas (46) and (47)). We want to add a few remarks.

1. A Hardy function $\phi(z)$ in the upper half plane

$$\mathbb{C}^+ := \{z = x + \mathrm{i}y \qquad y > 0\}$$

is a complex analytic function on $\mathbb{C}^+$ such that

$$\sup_{y>0} \int_{-\infty}^{\infty} |\phi(x + \mathrm{i}y)|^2 \, \mathrm{d}x = K < \infty.$$

The function $\phi(z)$ has boundary values on the real axis that determine a square integrable function $\phi(x)$ with

$$\int_{-\infty}^{\infty} |\phi(x)|^2 \, \mathrm{d}x \leqslant K.$$

A Hardy function on the upper half plane is uniquely determined by the function of its boundary values on the real axis [33–35]. The space of such functions is denoted by $\mathcal{H}_+^2$ and we have that $\mathcal{H}_+^2 \subset L^2(\mathbb{R})$. A similar definition goes for Hardy functions on the lower half plane. The space of these functions is denoted as $\mathcal{H}_-^2$. We have that [33–35]

$$\mathcal{H}_+^2 \oplus \mathcal{H}_-^2 = L^2(\mathbb{R}).$$

2. The algebra $\mathcal{A}_\pm$ is now isomorphic to $\mathcal{P} + \left(\mathcal{H}_\pm^2 \cap S\right) \otimes \left(\mathcal{H}_\mp^2 \cap S\right)$ and its product is defined as in (59). The topology in $\mathcal{H}_\pm^2 \cap S$ is inherited from S [15].

3. Let $O_{EE'} \in S \cap \mathcal{H}_+^2 \otimes S \cap \mathcal{H}_-^2$. Then

$$\mathrm{e}^{\mathrm{i}t(E-E')} \, O_{EE'} = \sum_{ij} \mathrm{e}^{\mathrm{i}tE} \varphi_i(E) \, \mathrm{e}^{-\mathrm{i}tE'} \psi_j(E').$$

If $t > 0$, $\mathrm{e}^{\mathrm{i}tE} \varphi_i(E) \in S \cap \mathcal{H}_+^2$, if $\varphi_i(E) \in S \cap \mathcal{H}_+^2$. Also, $\mathrm{e}^{-\mathrm{i}tE'} \psi_j(E') \in S \cap \mathcal{H}_-^2$, if $\psi_j(E') \in S \cap \mathcal{H}_-^2$ [15]. Both properties are true if and only if $t \geqslant 0$ [15].

Finally, let us prove that $O(O^\pm)$ commutes with $H_0(H)$, if and only if $O_{EE'} = 0$.

$$
\begin{aligned}
H_0 O &= \left[\int_0^\infty \mathrm{d}E \, E \, |E\rangle\langle E|\right] \left[\int_0^\infty \mathrm{d}E' \, O_{E'} \, |E'\rangle\langle E'|\right. \\
&\quad + \left.\int_0^\infty \mathrm{d}E' \int_0^\infty \mathrm{d}E'' \, O_{E'E''} \, |E'\rangle\langle E''|\right] \\
&= \int_0^\infty \mathrm{d}E \int_0^\infty \mathrm{d}E' \, E \, O_{E'} \, |E\rangle\langle E|E'\rangle\langle E'| \\
&\quad + \int_0^\infty \mathrm{d}E \int_0^\infty \mathrm{d}E' \int_0^\infty \mathrm{d}E'' \, E \, O_{E'E''} \, |E\rangle\langle E|E'\rangle\langle E''|.
\end{aligned}
\tag{65}
$$

Since $\langle E|E'\rangle = \delta(E - E')$, (65) finally gives

$$H_0 O = \int_0^\infty \mathrm{d}E \, E \, O_E \, |E\rangle\langle E| + \int_0^\infty \mathrm{d}E \int_0^\infty \mathrm{d}E'' \, E \, O_{EE''} \, |E\rangle\langle E''|.$$

We analogously prove that

$$
\begin{aligned}
O H_0 &= \int_0^\infty \mathrm{d}E \, E \, O_E \, |E\rangle\langle E| + \int_0^\infty \mathrm{d}E \int_0^\infty \mathrm{d}E'' \, E \, O_{E''E} \, |E''\rangle\langle E| \\
&= \int_0^\infty \mathrm{d}E \, E \, O_E \, |E\rangle\langle E| + \int_0^\infty \mathrm{d}E'' \int_0^\infty \mathrm{d}E \, E'' \, O_{EE''} \, |E\rangle\langle E''|.
\end{aligned}
\tag{66}
$$

Therefore, $H_0 O = O H_0$ if and only if $E O_{EE''} = E'' O_{EE''}$. This implies that $(E - E'') O_{EE''} = 0$ and since $O_{EE''}$ is nonsingular, we conclude that $O_{EE''} = 0$. Reciprocally, if $O_{EE''} = 0$, then, H_0 and O commute. Therefore, an operator O commutes with H_0 if and only if $O_{EE''} = 0$. As in the general case, $O_{EE''} \neq 0$, we conclude that $\mathcal{A}_0$ is a noncommutative algebra. The same result is obtained if we replace H_0 by H and $\mathcal{A}_0$ by $\mathcal{A}_{\pm}$.

References

[1] Laura R and Castagnino M 1998 Minimal irreversible quantum mechanics: the mixed states and the diagonal singularity *Phys. Rev.* A **57** 4140
[2] Laura R, Castagnino M and Id Betán R 1997 Perturbative method for generalized spectral decompositions *Physica* A **271** 357
[3] Laura R and Castagnino M 1998 Functional approach for quantum systems with continuous spectrum *Phys. Rev.* E **57** 3948
[4] Laura R and Castagnino M 2000 Functional approach to quantum decoherence and the classical final limit *Phys. Rev.* A **62** 2107
[5] van Hove L 1955 Energy corrections and persistent perturbation effects in continuous spectra *Physica* **21** 901
    van Hove L 1957 The approach to equilibrium in quantum statistics *Physica* **23** 441
    van Hove L 1959 The ergodic behaviour of quantum many-body sistems *Physica* **25** 268
[6] Antoniou I E, Laura R, Tasaki S and Suchanecki Z 1997 Intrinsic irreversibility of quantum systems with diagonal singularity *Physica* A **241** 737
    Antoniou I and Suchanecki Z 1997 Quantum systems with diagonal singularity *Adv. Chem. Phys.* **99** 299
[7] Petrosky T and Prigogine I 1998 Poincaré theorem and unitary transformation for classical and quantum systems *Physica* A **147** 439
[8] Petrosky T, Prigogine I and Tasaki S 1991 Quantum theory of nonintegrable systems *Physica* A **173** 175
[9] Gamow G 1928 Zur Quantentheorie des Atomkernes *Z. Phys.* **51** 204
[10] Bohm A 1979 *Quantum Mechanics Foundations and Applications* 1st edn (Berlin: Springer)
    Bohm A 1993 *Quantum Mechanics Foundations and Applications* 3rd edn (Berlin: Springer)
[11] Horwitz L and Marchand P 1971 The decay scattering system *Rocky Mt. J. Math.* **1** 225
[12] Antoniou I E and Prigogine I 1993 Intrinsic irreversibility and integrability of dynamics *Physica* A **192** 443
[13] Antoniou I E, Dmitrieva L, Kuperin Y and Melnikov Y 1997 Resonances and the extension of dynamics to rigged Hilbert space *Comput. Math. Appl.* **34** 399
[14] Antoniou I E and Melnikov Y 1998 Quantum scattering of resonances: poles of a continued *S*-matrix and poles of the extended resolvent *Irreversibility and Causality (Springer Lecture Notes in Physics vol 504)* ed A Bohm, H D Doebner and P Kielanowski (Berlin: Springer) pp 246–256
[15] Bohm A and Gadella M 1989 *Dirac Kets, Gamow Vectors and Gel'fand Triplets (Springer Lecture Notes in Physics vol 348)* (Berlin: Springer)
[16] Fonda L, Ghirardi G C and Rimini A 1978 Decay theory of unstable quantum systems *Rep. Prog. Phys.* **41** 587
[17] Antoniou I E, Gadella M and Suchanecki Z 1998 Some General Properties of the Liouville Operator *Irreversibility and Causality (Springer Lecture Notes in Physics vol 504)* ed A Bohm, H D Doebner and P Kielanowski (Berlin: Springer) pp 38–56
[18] Gadella M and Laura R 2001 Gamow dyads and expectation values *Int. J. Quantum Chem.* **81** 307
[19] Bratteli O and Robinson D W 1979 *Operator Algebras and Quantum Statistical Mechanics I, II* (Berlin: Springer)
[20] Castagnino M, Id Betán R, Laura R and Liotta R 2002 Quantum decay processes and Gamow states, to appear
[21] Amrein W, Jauch J and Sinha K 1977 *Scattering Theory in Quantum Mechanics* (Reading, MA: Benjamin)
[22] Gel'fand I M and Vilenkin N Ya 1964 *Generalized Functions vol 4* (New York: Academic)
[23] Maurin K 1968 *General Eigenfunction Expansion and Unitary Representations of Topological Groups* (Warsaw: Polish Scientific)
[24] Amrein W O, Martin Ph A and Misra B 1970 On the asymptotic condition of scattering theory *Helv. Phys. Acta* **43** 313
[25] Antoniou I E, Gadella M and Pronko G P 1998 Gamow vectors for degenerate scattering resonances *J. Math. Phys.* **39** 2459
[26] Bohm A, Antoniou I and Kielanowski P 1995 A quantum mechanical arrow of time and the semigroup time evolution of Gamow vectors *J. Math. Phys.* **36** 2593

[27] Gadella M and de la Madrid R 1994 Resonances and time reversal operator in rigged hilbert spaces *Int. J. Theor. Phys.* **38** 93

[28] Lee T D 1981 *Particle Physics and Introduction to Field Theory* (New York: Harwood Academic)

[29] Bohm A and Harshman N 1998 Quantum Theory in the Rigged Hilbert Space *Irreversibility from Causality in Irreversibility and Causality (Springer Lecture Notes in Physics vol 504)* ed A Bohm, H D Doebner and P Kielanowski (Berlin: Springer) pp 181–237

[30] Rudin W 1973 *Functional Analysis* (New York: McGraw-Hill)

[31] Pietsch A 1972 *Nuclear Locally Convex Spaces* (Berlin: Springer)

[32] Gelfand I M and Shilov G E 1968 *Generalized Functions* vol 2 (New York: Academic)

[33] Hoffmann K 1962 *Banach Spaces of Analytic Functions* (New Jersey: Prentice Hall)

[34] Duren P L 1970 *Theory of H^p Spaces* (New York: Academic)

[35] Koosis P 1990 *Introduction to H^p Spaces* (Cambridge: Cambridge University Press)
Koosis P 1990 *The Logaritmic Integral* (Cambridge: Cambridge University Press)